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Three Dimensional Periodic Systems with Trivial Dynamics, II

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0. INTRODUCTION

In this paper we study the competitive periodic system,

$$u_1' = f_1(t, u_1, u_2); \qquad u_2' = f_2(t, u_1, u_2, u_3); \qquad u_3' = f_3(t, u_2, u_3), \quad (0.1)$$

where $t \in \mathbb{R}$, $u = (u_1, u_2, u_3) \in \Omega$; $\Omega \subset \mathbb{R}^3$ is open and $F(t, u) := (f_i(t, u_1, u_2), f_2(t, u), f_3(t, u_2, u_3))$ is continuous in Ω, is T-periodic in the time t, and has continuous partial derivative $\partial F/\partial u$ defined on $\mathbb{R} \times \Omega$. We also assume that the following hypotheses hold.

(H_1) $\partial f_1/\partial u_2$, $\partial f_2/\partial u_1$, $\partial f_2/\partial u_3$, and $\partial f_3/\partial u_2 < 0$ on $\mathbb{R} \times \Omega$.

(H_2) F has a continuous extension $F_0\colon \mathbb{R} \times \Omega_0 \to \mathbb{R}^3$ such that $\partial F_0/\partial u$ is defined and continuous in a neighborhood Ω_0 of $\overline{\Omega}$.

In [5], it is proved that, in a certain sense, (0.1) has trivial dynamics. That is, if S is a solution of (0.1), defined and bounded in the interval $[\tau, \infty)$, then S is asymptotic as $t \to +\infty$, to a T-periodic solution of the system,

$$u' = F_0(t, u). \qquad (0.2)$$

Let H: $D_H \to \Omega$ be the Poincare map associated to (0.1). If the set Fix(H), of all fixed points of H, has an accumulation point in Ω and $F(t, u)$ is real analytic with respect to u, we shall show, in Section 2, that Fix(H) is an analytic curve M strictly linearly ordered with respect to the cone $\{u_1 > 0, u_2 < 0, u_3 > 0\}$. Moreover, if p is an endpoint of M, then $p \in \partial\Omega$. To show this theorem, we shall assume that f_i is decreasing with respect to x_i, for $i = 1, 3$.

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In Section 3 we apply the above results to obtain some properties about a competitor-competitor-mutualist model [4], while in Section 4, we study a model concerning a photoconductivity process [1, Sect. 6].

1. COMPARISON OF SOLUTIONS

In the following, $p \leq q$ (resp. $p < q$) denotes the usual order in $\mathbb{R}^3$ and $p \subseteq q$ (resp. $p \subset q$) denotes the order defined by $(-1)^{i-1} p_i \leq (-1)^{i-1} q_i$ (resp. $(-1)^{i-1} p_i < (-1)^{i-1} q_i$); $i = 1, 2, 3$. The domain of a (uncontinuable) solution u of (0.1) will be denoted by $\operatorname{dom}(u)$.

For reference purposes, we state the following basic comparison result [7].

1.1 PROPOSITION. *Let $u \neq v$ be solutions of* (0.1).

(a) *If $u(\tau) \leq v(\tau)$ for some τ, then $u(t) < v(t)$ in $(-\infty, \tau) \cap \operatorname{dom}(u) \cap \operatorname{dom}(v)$.*

(b) *If $u(\tau) \subseteq v(\tau)$ for some τ, then $u(t) \subset v(t)$ in $(\tau, \infty) \cap \operatorname{dom}(u) \cap \operatorname{dom}(v)$.*

As a particular case of Theorem 2.2 of [5], we have:

1.2 THEOREM. *If $S\colon [0, \infty) \to \Omega$ is a bounded solution of Eq.* (0.1) *then there exists a T-periodic solution $S\colon \mathbb{R} \to \overline{\Omega}$ of* (0.2) *such that $S(t) - S_*(t) \to 0$ as $t \to +\infty$.*

2. ON THE STRUCTURE OF THE SET OF FIXED POINTS OF H

We begin with the following result.

2.1 PROPOSITION. *Let $u = (u_1, u_2, u_3) \neq v = (v_1, v_2, v_3)$ be T-periodic solution of* (0.1). *Then $u_i(t) \neq v_i(t)$ for all $t \in \mathbb{R}$ and $i = 1, 2, 3$.*

Proof. Suppose that $u_1(\tau) = v_1(\tau)$ for some τ and let us consider the following two possibilities.

(P_1) $u_2(\tau) \leq v_2(\tau)$ and $u_3(\tau) \leq v_3(\tau)$. By Proposition 1.1, $u_1(t) < v_1(t)$ for $t < \tau$, and so, $u_1(\tau) = u_1(\tau - T) < v_1(\tau - T) = v_1(\tau)$, a contradiction.

(P_2) $u_2(\tau) \leq v_2(\tau)$ and $u_3(\tau) \geq v_3(\tau)$. By Proposition 1.1, $u_1(t) > v_1(t)$ for $t < \tau$, and as above, we obtain a contradiction.

Similar contradictions are obtained in the other cases, and the proof follows easily.

In the next, we assume that the following hypothesis holds.

(H_3) f_i is decreasing with respect to x_i; $i = 1, 3$.

2.2 THEOREM. *Let $u \neq v$ be T-periodic solutions of* (0.1). *Then, either $u \subset v$ or $v \subset u$. That is, $u(t) \subset v(t)$ for all $t \in \mathbb{R}$ or $v(t) \subset u(t)$ for all $t \in \mathbb{R}$.*

Proof. By Proposition 2.1, we can assume $u_1(t) < v_1(t)$ for all $t \in \mathbb{R}$. Suppose now that $u_2(t) < v_2(t)$; $t \in \mathbb{R}$; then, by (H_1) and (H_3), we have $u_1'(t) = f_1(t, u_1(t), u_2(t)) > f_1(t, u_1(t), v_2(t)) \geq f_1(t, v_1(t), v_2(t)) = v_1'(t)$, and thus, $u_1 - v_1$ is not periodic. This contradiction and Proposition 2.1 imply $u_2 > v_2$.

Analogously, if $u_3(t) > v_3(t)$; $t \in \mathbb{R}$; then $(u_3 - v_3)' > 0$ and this contradition completes the proof.

2.3 COROLLARY. *Assume $\Omega = \mathbb{R}^3$. If $F(t, u)$ is linear in u, then the linear space of T-periodic solutions of* (0.1) *has at most dimension one. That is, if $\Phi(t)$ denotes the fundamental matrix of* (0.1) *with $\Phi(0) = I$, then* $\operatorname{rank}(I - \Phi(T)) \geq 2$.

Proof. If $u \neq 0$ is a T-periodic solution of (0.1) then, by Theorem 2.2, either $u(t) \subset 0$ for all t or $0 \subset u(t)$ for all t.

On the other hand, if $P \subset \mathbb{R}^3$ is a two-dimensional linear subspace, then there exists $p = (p_1, p_2, p_3) \in P$, $p \neq 0$, such that $p_1 = 0$; and the proof follows easily.

Corollary 2.3 is also a consequence of Theorem 1.3 of [5].

Using the arguments in [3], we can prove the following version of a result of Nakajima and Seifert [2].

2.4 LEMMA. *Let $A: W \to \mathbb{R}^n$ be a real analytic function defined in an open subset of W of $\mathbb{R}^n$ and suppose that $A^{-1}(0)$ has an accumulation point $p \in W$. If* $\operatorname{rank}(A'(p)) = n - 1$, *then there exists an open set $U \subset W$ containing p, such that $U \cap A^{-1}(0)$ is a one-dimensional real analytic submanifold of $\mathbb{R}^n$.*

In order to state the main result of this section, we need some notations and definitions. Given $x \in \Omega$, $u(t, x)$ denotes the solution of (0.1) determined by the initial condition $u(0, x) = x$. We also recall the Poincare map $H: D_H \to \Omega$; $H(x) = u(T, x)$; where $D_H := \{x \in \Omega: u(\cdot, x)$ is defined in $[0,T]\}$.

We say that a nonempty subset M of $\mathbb{R}^3$ is $\subset$-linearly ordered if either $x \subset y$ or $y \subset x$, for all $x, y \in M$, $x \neq y$. In Theorem 2.2, we have proved that $\operatorname{Fix}(H)$ is $\subset$-linearly ordered. We say that M is $\subset$-left bounded (resp. $\subset$-right bounded) if there exists $p \in \mathbb{R}^3$ such that, $p \subseteq x$ (resp. $x \subseteq p$) for all $x \in M$. In this case, we write $p \subseteq M$ (resp. $M \subseteq p$)

and we say that p is a $\subset$-lower (resp. $\subset$-upper) bound of M. We also define $\inf(M) = (\inf(\pi_1(M)), \sup(\pi_2(M)), \inf(\pi_3(M)))$ (resp. $\sup(M) = (\sup(\pi_1(M)), \inf(\pi_2(M)), \sup(\pi_3(M)))$), where from now on, $\pi_i\colon \mathbb{R}^3 \to \mathbb{R}$ denotes the natural projection, $\pi_i(u) = u_i$. Note that $p \subseteq \inf(M)$ if p is a $\subset$-lower bound of M.

We say that $W \subset \mathbb{R}^3$ is $\subset$-convex if, given $x, y \in W$; $x \subset y$; the set $\{z \in \mathbb{R}^3 \colon x \subseteq z \subseteq y\}$ is contained in W.

2.5 THEOREM. *Suppose that Ω is $\subset$-convex and that $F(t,u)$ is real analytic with respect to u. If $M := \mathrm{Fix}(H)$ has an accumulation point $p \in \Omega$, then M is a connected real analytic submanifold of $\mathbb{R}^3$ of dimension one. Moreover, if M is $\subset$-left (resp. $\subset$-right) bounded then* $\inf(M) \in \partial\Omega$ (*resp.* $\sup(M) \in \partial\Omega$).

Proof. By Proposition 2.1, the restriction $\rho\colon M \to J := \pi_1(M)$ is bijective. We shall prove that ρ^{-1} is continuous. To this end, let $\{x^k = (x_1^k, x_2^k, x_3^k)\}$ be a sequence in M such that $\{x_1^k\}$ converges to $\rho(x)$ for some $x = (x_1, x_2, x_3) \in M$ and define $A = \{k \in \mathbb{N} \colon x_1^k \geq x_1\}$, $B = \{k \in \mathbb{N} \colon x_1^k \leq x_1\}$.

Claim. If A is infinite, then $x^k \to x$ as $k \to +\infty$; $k \in A$. To show this, it suffices to prove that each subsequence $\{y^m\}$ of $\{x^k \colon k \in A\}$ has a subsequence converging to x. Since $y_1^m \geq x_1$, $\{y^m\}$ has a subsequence $\{z^r\}$ such that $\{z_1^r\}$ is decreasing. From this and Theorem 2.2, $\{z_2^r\}$ is increasing and $\{z_3^r\}$ is decreasing. By Theorem 2.2 once again, $x \subseteq z^r$ for all $r \in \mathbb{N}$ and thus, there exists $z_2, z_3 \in \mathbb{R}$ such that $z_i^r \to z_i$ as $r \to +\infty$, for $i = 2, 3$. Let us define $z = (x_1, z_2, z_3)$. Then, $x \subseteq z \subseteq z^r$ and hence, $z \in \Omega$ since Ω is $\subset$-convex. From this, $z \in \mathrm{Fix}(H)$ and, by Proposition 2.1, $z = x$. The proof of the Claim is complete.

Analogously, if B is infinite, then $x^k \to x$ as $k \to +\infty$, $k \in B$. From this, $\{x^k\}$ converges to x, and hence ρ^{-1} is continuous.

By Corollary 2.3, $\mathrm{rank}(I - H'(p)) = 2$, since p is an accumulation point of M, and by Lemma 2.4, J has a nontrivial component J_0. Note that, by Lemma 2.4 once again, J_0 is open. We claim that $M = M_0 := \rho^{-1}(J_0)$. To show this assume, on the contrary, that there exists $q \in M$ such that $q \notin M_0$. Since M_0 is connected (ρ is a homeomorphism), then by Theorem 2.2, we can suppose that $q \subseteq x$ for all $x \in M_0$. That is, M_0 is $\subset$-left bounded. From this, $q \subseteq p_0 := \inf(M_0) \subseteq x$ for $x \in M_0$, and so, $p_0 \in \Omega$. Consequently, $p_0 \in \mathrm{Fix}(H)$. On the other hand, $\rho(p_0)$ is the left endpoint of J_0 and thus, $J_1 := J_0 \cup \{\rho(p_0)\}$ is an interval. If J_2 denotes the component of J containing J_1 then $J_0 \subset J_2 \neq J_0$, and this contradiction proves that $M = M_0$. Thus, M is connected and by Lemma 2.4, M is a real analytic submanifold of $\mathbb{R}^3$ of dimension one.

Assume now that M is $\subset$-left bounded. If $\inf(M) \in \Omega$ then, by Lemma 2.4, $\inf(M)$ is an interior point of M. This contradiction proves that $\inf(M) \in \partial\Omega$. Analogously, $\sup(M) \in \partial\Omega$ is M is $\subset$-right bounded, and the proof is complete.

The arguments in this section can be used to improve the main result in [3].

3. A COMPETITOR-COMPETITOR-MUTUALIST MODEL

In this section we study the periodic Kolmogorov system

$$\begin{aligned} u_1' &= u_1 F_1(t, u_1, u_2) \\ u_2' &= u_2 F_2(t, u_1, u_2, u_3) \\ u_3' &= u_3 F_3(t, u_2, u_3), \end{aligned} \tag{3.1}$$

where $t \in \mathbb{R}$, $u = (u_1, u_2, u_3) \in \mathbb{R}_+^3 :=$ the first nonnegative octant of $\mathbb{R}^3$, and $F\colon \mathbb{R} \times \mathbb{R}_+^3 \to \mathbb{R}^3$ is a continuous function of the form $F(t,u) = (F_1(t, u_1, u_2), F_2(t, u), F_3(t, u_2, u_3))$, which is T-periodic in the time t, and has continuous partial derivative $\partial F/\partial u$ on $\mathbb{R} \times \mathbb{R}_+^3$. We shall assume that:

(K$_1$) $\partial F_1/\partial u_1$, $\partial F_1/\partial u_2$, $\partial F_2/\partial u_1$, $\partial F_2/\partial u_2$, $\partial F_3/\partial u_3 < 0$ on $\mathbb{R} \times \mathbb{R}_+^3$ and $\partial F_3/\partial u_2$, $\partial F_3/\partial u_2 > 0$ on $\mathbb{R} \times P^3$; where $P^3 := \inf(\mathbb{R}_+^3)$.

(K$_2$) $\int_0^T F(t,0)\, dt > 0$ (here, $>$ is the usual order of $\mathbb{R}^3$).

(K$_3$) There exists $R > 0$ such that $\int_0^T F_1(t, R, 0)\, dt < 0$, $\int_0^T F_2(t, 0, R, 0)\, dt < 0$, and $\int_0^T F_3(t, 0, R)\, dt < 0$.

(K$_4$) $F_2(t,u) = G(t, u_2) + u_1 F^0(t,u)$, where $G(t,y)$ is continuous, T-periodic in T, and has continuous partial derivation $\partial G/\partial y < 0$ on $\mathbb{R} \times (0, \infty)$.

The above system is a model for a three species process which includes a competitor u_1, a mutualist-competitor u_2, and a mutualist u_3 that decreases the effect of u_1 on u_2 ($\partial F_2/\partial F_3 > 0$). In the absence of the mutualist ($u_3 = 0$), system (3.1) becomes a competitive system ($\partial F_1/\partial u_2 < 0$, $\partial F_2/\partial u_1 < 0$), while in the absence of the competitor u_1, system (3.1) is reduced to a mutualist system ($\partial F_2/\partial u_3 > 0$, $\partial F_3/\partial u_2 > 0$). For details, see [4]. Condition (K$_2$) says that each species, in the absence of the other ones, has a positive average growth rate. Finally, condition (K$_3$) will be used in order to ensure that the system is dissipative.

By Corollary 2 of [9], the equation

$$x' = xF_3(t, 0, x) \tag{3.2}$$

has a unique T-periodic positive solution that we denote by U_3. Moreover, U_3 is globally asymptotically stable. That is, if w is a solution of (3.2) and $w(\tau) > 0$, for some τ, then w is defined on $[\tau, \infty)$ and $w(t) - U_3(t) \to 0$ as $t \to +\infty$.

By the same argument, the equation $x' = xF_1(t, x, 0)$ (resp. $x' = xF_2(t, 0, x, 0) = xG(t, x)$) has a unique positive T-periodic solution that we denote by U_1 (resp. U_2). Moreover, U_1 (resp. U_2) is globally asymptotically stable.

In the following, we also assume:

(K_5) $\int_0^T F_3(t, U_2(t), S)\, dt < 0$ for some $S \geq R$.

By (K_1), $F_3(t, U_2(t), 0) > F_3(t, 0, 0)$ and by Corollary 2 of [9] once again, the equation $x' = xF_3(t, U_2(t), x)$ has a unique positive T-periodic solution which we denote by U_3^*.

In the next, $u = (u_1, u_2, u_3)$ denotes an uncontinuable solution of (3.1) such that $u(0) > 0$. It is easy to show that $u(t) > 0$ for all t in the domain of u. For this reason, we say that u is positive.

3.1 Proposition. *u is defined and bounded on $[0, \infty)$.*

Proof. Let us define $I = [0, \infty) \cap$ (domain of u) and let v_1 be the solution of the problem $x' = xF_1(t, x, 0)$, $x(0) = u_1(0)$, then $u_1(t) \leq v(t)$; $t \in I$; since, by (K_1), $F_1(t, x, 0) > F_1(t, x, u_2(t))$. Analogously, $u_2(t) \leq v_2(t)$; $t \in I$; where v_2 denotes the solution of the problem $x' = xG(t, x)$, $x(0) = u_2(0)$. Note that $G(t, x) = F_2(t, 0, x, u_3(t)) > F_2(t, u_1(t), x, u_3(t))$.

Let v_3 be the solution of the problem $x' = xF_3(t, v_2(t), x)$, $x(0) = u_3(0)$. Since $v_2(t) - U_2(t) \to 0$ as $t \to +\infty$, there exists a positive constant M such that $v_2(t) \leq M$, for all $t \geq 0$. In particular, $\ln[v_3(t)/v_3(0)] \leq \int_0^t F_3(s, M, 0)\, ds$, if $t > 0$ belongs to the domain of v_3. Thus, v_3 is defined on $[0, \infty)$.

Claim. $v_3(t) - U_3^*(t) \to 0$ as $t \to +\infty$. To prove this, (K_5) there is a decreasing sequence $\{\varepsilon_n\}$ in $(0, \infty)$ converging to zero such that $\int_0^T F_2(t, U_2(t) + \varepsilon_n, S)\, dt < 0$, for all $n \in \mathbb{N}$. In particular, the equation

$$x' = xF_3(t, U_2(t) + \varepsilon_n, x) \tag{3.3}$$

has a positive T-periodic solution U^n which is globally asymptotically stable.

Now, let us fix $t_n > 0$ such that $v_2(t) \leq U_2(t) + \varepsilon_n$ for $t \geq t_n$, and define v^n as the solution of (3.3) determined by the initial condition

$x(t_n) = v_3(t_n)$. Then,

$$v_3(t) \le v^n(t) \qquad \text{for } t \ge t_n. \tag{3.4}$$

Remember also that

$$v^n(t) - U^n(t) \to 0 \qquad \text{as } t \to +\infty. \tag{3.5}$$

Since $\{\varepsilon_n\}$ is decreasing then, by Proposition 1.3 of [8], $U^1 \ge U^2 \ge \cdots \ge U_3^*$. In particular, $\{U^n(t)\}$ has a finite limit $W(t)$ for all $t \in \mathbb{R}$. Since $\{U^n\}$ is bounded, the same holds for the sequence of derivatives $\{(U^n)'\}$ and by Ascoli's theorem, $\{U^n\}$ converges to W uniformly on $\mathbb{R}$. Hence, $\{(U^n)'\}$ converges uniformly to $WF_3(t, U_2, W)$ and thus, W is a positive T-periodic solution of (3.3). Consequently, $W = U_3^*$ and so, $\{U^n\}$ converges uniformly to U_3^*.

Given $\varepsilon > 0$, let us fix an integer $N \in \mathbb{N}$ such that $U^N(t) \le U_3^*(t) + \varepsilon$ for all $t \in \mathbb{R}$. By (3.5), we can also fix $\tau_N \ge t_N$ such that $v^N(t) \le U^N(t) + \varepsilon$ for $t \ge \tau_N$. From this and (3.4), $v_3(t) \le U_3^*(t) + 2\varepsilon$ for $t \ge \tau_N$. Analogously, there exists $\sigma_N \ge t_N$ such that $v_3(t) \ge U_3^*(t) - 2\varepsilon$ for $t \ge \sigma_N$, and the proof of the claim is complete.

Finally, $F_3(t, u_2(t), x) \le F_3(t, v_2(t), x)$, and hence, $u_3(t) \le v_3(t)$; $t \in I$. The proof follows now easily, since $v_1(t) - U_1(t) \to 0$ as $t \to +\infty$.

3.2 COROLLARY. *Let u be as above. Then,* (3.1) *has a T-periodic solution $v = (v_1, v_2, v_3)$ such that $u(t) - v(t) \to 0$ as $t \to +\infty$. Moreover, $v_3 \ge U_3$.*

Proof. It is easy to show that that, by the change of variables $(u_1, u_2, u_3) \to (u_1, u_2, -u_3)$, the system (3.1) becomes a competitive system that satisfies the assumptions in Theorem 1.2. Thus, there exists a T-periodic solution $v = (v_1, v_2, v_3)$ such that $u(t) - v(t) \to 0$ as $t \to +\infty$.

Now, let w be the solution to (3.2) such that $w(0) = u_3(0)$. Since $F_3(t, u_2(t), z) > F_3(t, 0, z)$, for $z > 0$, then $w < u_3$ on $(0, \infty)$ and the proof follows easily because $w(t) - U_3(t) \to 0$ as $t \to +\infty$.

In the following we define nonnegative T-periodic solutions of (3.1) by $\mathbb{U}_0 = (0, 0, U_3)$, $\mathbb{U}_1 = (U_1, 0, U_3)$, $\mathbb{U}_2 = (0, U_2, U_3^*)$. We also denote by H: $P^3 \to \mathbb{R}^3$ the Poincare map of (3.1). By Proposition 3.1, H is well defined.

3.3 PROPOSITION. *Let $v \not\equiv 0$ be a nonnegative and nonpositive T-periodic solution of* (3.1), *such that $v_3 > 0$, then $v = \mathbb{U}_i$ for some $i = 0, 1, 2$.*

Proof. If $v_1(\tau) = 0$ for some τ, then $v_1 \equiv 0$ and so (v_2, v_3) is a T-periodic solution of the system

$$\begin{aligned} y' &= yG(t, y) \\ z' &= zF_3(t, y, z). \end{aligned} \tag{3.6}$$

By (3.6), either $v_2 \equiv 0$ or $v_2 \equiv U_2$ and hence, either $v = \mathbb{U}_0$ or $v = \mathbb{U}_2$.

Assume now that $v_2(\tau) = 0$ for some τ. Then, $v_2 \equiv 0$ and so v_3 is a positive T-periodic solution of (3.2). Thus, $v_3 = U_3$. On the other hand, v_1 is a T-periodic solution of the logistic equation $x' = xF_1(t, x, 0)$ and so either $v_1 = 0$ or $v_1 = U_1$. That is, either $v = \mathbb{U}_0$ or $v = \mathbb{U}_1$, and the proof is complete.

3.4 PROPOSITION. (a) *Let* $v = (v_1, v_2, v_3)$ *be a T-periodic positive solution of* (3.1) *such that* $v_3 > 0$. *Then,* $v_1 < U_1$, $v_2 < U_2$, *and* $U_3 < v_3 < U_3^*$. *In particular,* $\mathrm{Fix}(H)$ *is bounded.*

(b) $\mathbb{U}_0(0)$ *is not an accumulation point of* $\mathrm{Fix}(H)$.

(c) *If* $p \in \partial\mathbb{R}^3_+$ *is an accumulation point of* $\mathrm{Fix}(H)$, *then either* $p = \mathbb{U}_1(0)$ *or* $p = \mathbb{U}_2(0)$.

Proof. (a) Since $F_1(t, x, 0) > F_1(t, x, v_2(t))$ then $v_1 < U_1$. See the argument in Proposition 1.3 of [8]. On the other hand, $G(t, x) = F_2(t, 0, x, v_3(t)) > F_2(t, v_1(t), x, v_3(t))$ and hence, $v_2 < U_2$. From this, $F_3(t, 0, x) < F_3(t, v_2(t), x) < F_3(t, U_2(t), x)$, and so $U_3 < v_3 < U_3^*$.

(b) Suppose that there exists a sequence $\{p^k\}$ in $\mathrm{Fix}(H)$ such that $p^k \to \mathbb{U}_0(0)$, and define $u^k(t) = u(t, p^k)$. Then, $u^k(t) \to \mathbb{U}_0(t)$ uniformly on $\mathbb{R}$ and by (3.1), $\int_0^T F_1(t, u_1^k(t), u_2^k(t))\,dt = 0$, since u^k is T-periodic. Hence, $\int_0^T F_1(t, 0, 0)\,dt = 0$, which contradicts (K_1) and proves (b).

(c) Let $\{p^k\}$ be a sequence in $\mathrm{Fix}(H)$ converging to p, let u^k be as above, and define $v(t) = u(t, p)$. By part (a), $v_3 \geq U_3$ since $u^k(t) \to v(t)$, and by Proposition 3.3, $v = \mathbb{U}_i$ for some $i = 0, 1, 2$, since $p \in \partial\mathbb{R}^3_+$. The proof follows now from part (b).

3.5 THEOREM. *In addition to* (K_1)–(K_5) *assume that F has a continuous extension* $F_0(t, u)$ *to* $\mathbb{R} \times \Gamma$ *which is analytic in* $u \in \Gamma$, *where* Γ *is an open set containing* $\mathbb{R}^3_+$. *If* $M := \mathrm{Fix}(H)$ *is infinite, then M is a bounded connected real analytic submanifold of* $\mathbb{R}^3$ *of dimension one which is linearly ordered with respect to the cone* $\{x_1 > 0, x_2 < 0, x_3 < 0\}$. *Moreover,* $\mathbb{U}_1(0)$ *and* $\mathbb{U}_2(0)$ *are the endpoints of M.*

Proof. We first prove that the result is true if M has an accumulation point $q \in P^3$, and then we show that this assumption holds.

By the change of variables $\Phi(x_1, x_2, x_3) = (x_1, x_3, -x_3)$, the system (3.1) becomes a competitive system that satisfies the assumption in Theorem 2.5, with $\Omega = \Phi(P^3)$ and $p = \Phi(q)$. This, $\Phi(M)$ is a bounded connected real analytic submanifold of dimension one, and its endpoints belong to $\partial\Omega$. By Proposition 3.4, we conclude that $\sup(\Phi(M)) = \Phi(\mathbb{U}_1(0))$ and $\inf(\Phi(M)) = \Phi(\mathbb{U}_2(0))$, and the proof will be complete if we show that M has a positive accumulation point.

By Proposition 3.4, M is bounded and hence, M has an accumulation point p in $\mathbb{R}_+^3$. If $p \in P^3$ there is nothing to prove. Assume now that $p \in \partial\mathbb{R}_+^3$ and remember that, by Proposition 3.4, $p = \mathbb{U}_i(0)$ for some $i = 1, 2$.

Let H_0 be the Poincare map of system $u' = F_0(t, u)$. Obviously, H_0 is an extension of H, defined on a neighborhood of R_+^3 and hence, rank$(I - H_0'(p)) \leq 2$, since p is an assumulation point of Fix(H_0).

Assume now that $p = \mathbb{U}_2(0)$. By definition of U_2 and U_3^*, the linearized system of (3.1) at $\mathbb{U}_2$ can be written as

$$\begin{aligned} x_1' &= x_1 F_1\big(t, 0, U_2(t)\big) \\ (x_2/U_2)' &= (\partial F_2/\partial u_1)\big(t, \mathbb{U}_2(t)\big)x_1 + (\partial F_2/\partial u_2)\big(t, \mathbb{U}_2(t)\big)x_2 \\ (x/U_3^*)' &= (\partial F_3/\partial u_2)\big(t, U_2(t), U_3^*(t)\big)x_2 \\ &\quad + (\partial F_3/\partial u_3)(t, U_2^*, U_3)/\partial u_3 x_3 \end{aligned} \tag{3.7}$$

since, by (K_4), $(\partial F_2/\partial u_3)(0, x_2, x_3) = 0$ for all $x_2, x_3 \geq 0$.

It is well known that $H_0'(\mathbb{U}_2(0)) = \Psi(T)$, where $\Psi(t)$ is the fundamental matrix of (3.7), with $\Psi(0) =$ identity. On the other hand the matrix $\Psi(T)$ is given by

$$\begin{matrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & a & \lambda_3 \end{matrix} \tag{3.8}$$

where $\lambda_2 = \exp(\int_0^T (\partial F_2/\partial u_2)(s, \mathbb{U}(s))U_2(s)\,ds) \in (0, 1)$ and

$$\lambda_3 = \exp\left(\int_0^T (\partial F_3/\partial u_3)\big(s, U_2(s), U_3^*(s)\big)U_3^*(s)\,ds\right) \in (0, 1).$$

In particular, rank$(I - H_0'(\mathbb{U}_2(0))) = 2$.

By Lemma 2.4, there exists $\varepsilon > 0$ and an analytic function $y = (y_1, y_2, y_3)\colon (-\varepsilon, \varepsilon) \to \mathrm{Fix}(H_0)$ such that $y(0) = \mathbb{U}_2(0)$ and $y'(0) \neq 0$. In particular, we can assume that $y_i(s) > 0$ for $i = 2, 3$ and $s \in (-\varepsilon, \varepsilon)$.

Claim. $y_1(s_0) > 0$ for some s_0. To show this assume, on the contrary, that $y_1(s) \leq 0$ for all $s \in (-\varepsilon, \varepsilon)$. Since $y_1(0) = 0$, we obtain $y_1'(0) = 0$. On the other hand, from the relation $y(s) = H_0(y(s))$, we have $y'(0) = \Psi(T)y'(0)$, and by (3.8)

$$\lambda_2 y_2'(0) = y_2'(0); \qquad a y_2'(0) + \lambda_3 y_3'(0) = y_3'(0).$$

From this, $y_2'(0) = y_3'(0) = 0$ and this contradiction $(y'(0) \neq 0)$ proves the claim.

By the above Claim, $y(s_0) > 0$ is an accumulation point of M and the proof of this case is complete. Analogously, if $p = \mathbb{U}_1(0)$, we show that M has an assumulation point in P^3, and the proof is complete.

4. PHOTOCONDUCTIVITY

In [1, Sect. 6] the following system

$$\begin{aligned} x' &= G - \alpha x(Y - y) - cx + \gamma y \\ y' &= \alpha x(Y - y) - \delta yz - \gamma y \\ z' &= G - \delta yz - dz \end{aligned} \tag{4.1}$$

is proposed as a mathematical model for a photoconductivity process in intrinsic semiconductors. Here, G, α, Y, c, γ, δ, d are positive constants.

In this section, we consider the periodic system

$$x' = f(t, x, y), \qquad y' = g(t, x, y, z), \qquad z' = h(t, y, z), \tag{4.2}$$

where f, g, h satisfy the relevant properties of system (4.1).

More precisely, the function $F\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$, $F(t, x, y, z) = (f(t, x, y), g(t, x, y, z), h(t, y, z))$ is continuous, is T-periodic in t, and has continuous partial derivatives F_x, F_y, F_z in $\mathbb{R} \times \mathbb{R}^3$. We also assume that:

(L_1) $f(t, 0, y), g(t, x, 0, y), h(t, y, 0) > 0$ for $y, z \geq 0$ and $t \in \mathbb{R}$.

(L_2) There exist $X, Y, Z > 0$ such that:

$h(t, y, z) < 0$ if $y > 0$ and $z > Z$.

$g(t, x, y, z) \leq G(t, y, z) < 0$ if $x, z \geq 0$ and $y \geq Y$; where G is continuous in $\mathbb{R} \times [Y, \infty) \times [0, \infty)$ and T-periodic in t.

$f(t, x, y) < 0$ if $0 < y < Y$ and $x > X$.

(L_3) For each $R > 0$ there exist $A, B > 0$ such that $f(t, x, y) \leq A + Bx$ for $0 \leq y \leq R$.

(L_4) $f_x > 0 > f_y$ in $(0, X) \times (0, Y)$. $g_x > 0$, $g_y < 0$, $g_z < 0$ on $(0, X) \times (0, Y) \times (0, Z)$. $h_y < 0, h_z < 0$ on $(0, Y) \times (0, Z)$.

In the following, $S = (u, v, w)$ denotes a solution of (4.2) such that $S(0) \geq 0$. The maximum domain of this solution will be denoted by D.

4.1. Remarks. (a) Using (L_1), it is easy to show that $S(t) > 0$, for $t \in D \cap (0, \infty)$. In particular, if S is a nonnegative T-periodic solution of (4.2), then S is positive.

(b) Analogously, using (L_2) it is easy to show that

$$w(t) \leq \max\{w(\tau), Z\}$$

and (4.3)

$$v(t) \leq \max\{v(\tau), Y\}; \qquad \text{for } t, \tau \in D,\ t \geq \tau \geq 0.$$

4.2 PROPOSITION. (a) *S is defined on* $[0, \infty)$.

(b) *There exists $s \geq 0$ such that $v(s) < Y$.*

(c) *If $v(s) \leq Y$ for some $s \geq 0$ then, $u(t) \leq \max\{u(s), X\}$ for $t \geq s$. In particular, u is bounded on $[0, \infty)$ and by* (4.3), *the same holds for S.*

(d) *If $v(s) \leq Y$ for some $s \geq 0$, then $v(t) < Y$ for $t > s$.*

Proof. Let us write $[0, \omega) = D \cap [0, \infty)$ and $R = \max\{v(0), Y\}$. By (4.3), $v \leq R$ on $[0, \omega)$, and by (L_3) there exist $A, B > 0$ such that $u'(t) \leq A + Bu(t)$ in $[0, \omega)$. Assume now that $\omega < +\infty$. Then u is bounded in $[0, \omega)$ and by (4.3), S is bounded in this interval. This contradiction proves (a).

To show (b), let us assume that $v(t) > Y$ for $t \geq 0$. Then, by (4.3), v is decreasing in $[0, \infty)$ and so $v(t) \to \lambda$ as $t \to +\infty$ for some $\lambda \geq Y$. In particular, there exists a sequence $t_n \to +\infty$ such that $v'(t_n) \to 0$. On the other hand, w is bounded on $[0, \infty)$; see (4.2); and so, we can suppose that $w(t_n)$ converges to some $\eta \geq 0$. From this, (4.2), and (L_2), $0 \leq G(r, \lambda, \eta) < 0$; for some $r \in [0, T]$; and this contradiction proves (b).

To prove (c), let us define $M = \max\{u(s), X\}$ and suppose that our claim is false. Since $u(a) \leq M$, there exists $t_0 > s$ such that $u(t_0) > M$ and $u'(t_0) > 0$. On the other hand, by (4.3), $0 \leq v(t_0) \leq \max\{v(s), Y\} = Y$, and by (L_2), $f(t_0, u(t_0), v(t_0)) \leq 0$, since $u(t_0) > X$. This contradiction proves (c).

To show (d), we remark that, by (4.3), $v(t) \leq Y$ for $t > s$. Assume now that $v(t) = Y$, for some $t > s$. Then, $0 = v'(t) = g(t, u(t), Y, w(t)) \leq G(t, Y, w(t)) < 0$, and this contradiction ends the proof.

4.3 PROPOSITION. *Let us define $u_M = \limsup_{t \to +\infty} u(t)$ and v_M, w_M analogously. Then, $u_M < X, v_M < Y$, and $w_M < Z$.*

Proof. By Lemma 1.2 of [6], there exists a sequence $t_n \to +\infty$ such that $v(t_n) \to v_M$ and $v'(t_n) \to 0$. On the other hand, u, w are bounded on $[0, \infty)$ and thus we can assume, without loss of generality, that $u(t_n) \to \lambda$ and $w(t_n) \to \eta$. From this, $0 = g(r, \lambda, v_M, \eta) \leq G(r, v_M, \eta)$, for some r, and by (L_2), $v_M < Y$.

By the above argument, there exist $r \in \mathbb{R}$, $\mu \in [0, Y)$ such that $0 = f(r, u_M, \mu)$, and by (L_4), $0 < f(r, u_M, Y)$. Hence, $u_M < X$ since (L_2) holds. The rest of the proof is analogous.

4.4 THEOREM. *Assume* (L_1)–(L_4) *holds.*

(a) *If* S *is a solution of* (4.2) *such that* $S(0) \geq 0$ *then* (4.2) *has a positive* T*-periodic solution* S^0 *such that* $S(t) - S^0(t) \to 0$ *as* $t \to +\infty$.

(b) *If* $F(t, u)$ *is analytic in* u, *then the set of all positive* T*-periodic solutions of* (4.2) *is finite.*

Proof. By the change of variables $(u, v, w) \to (u, -v, -w)$, system (4.2) becomes a competitive system on $\Omega = \{(x, y, z): 0 < x < X, -Y < y < 0, -Z < z < 0\}$ and (a) follows from Theorem 1.2 and Proposition 4.3.

Assume now that $F(t, u)$ is analytic in u. By Remark 4.1 and Proposition 4.3, we know that the competitive system obtained above has no T-periodic solutions in $\partial\Omega$. The proof follows not from Theorem 2.5, since Ω is bounded and $\subset$-convex.

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